Non-Commutative Rings and their Applications

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Modules with the Exchange Property

Joint work with Yasser Ibrahim of both Taibah and Cairo Universities

A right R-module M is said to satisfy the (full) exchange property if for any two direct sum decompositions $M \oplus N = \oplus_{i \in I} N_i$, there exist submodules $K_i \subseteq N_i$ such that $M \oplus N = M \oplus (\oplus_{i \in I} K_i)$. If this holds only for $|I| < \infty$, then M is said to satisfy the finite exchange property.

A ring R for which R_R has the finite exchange property is called an exchange ring.

It is an open question due to Crawley and Jónsson whether the finite exchange property always implies the full exchange property.

The exchange property is of importance because it provides a way to build isomorphic refinements of different direct sum decompositions, which is precisely what is needed to prove the famous Krull-Schmidt-Remak-Azumaya Theorem.

This question was provided a positive answer for:

1. Quasi-injective modules by L. Fuchs, where a module M is quasi-injective if it is invariant under endomorphisms of its injective hull.

2. Quasi-continuous modules by Mohamed and Müller & Oshiro and Rizvi, where a module M is quasi-continuous if it is invariant under idempotent-endomorphisms of its injective hull.

3. Auto-invariant Modules by P. Guil Assensio and A. Srivastava, where a module M is called auto-invariant if it is invariant under automorphisms of its injective hull.

4. Square-free modules by P. Nielsen, where M is called square-free if it does not contain a submodule isomorphic to a square $A \oplus A$.

A result of Warfield asserts that a module M_R has the finite exchange property iff $End(M_R)$ is an exchange ring. The notion of exchange rings is left-right symmetric, indeed, Nicholson showed that a ring R is an exchange ring iff idempotents lift modulo every right ideal of R, iff idempotents lift modulo every left ideal of R.

Exchange rings are closely related to another interesting class of rings called clean rings that was first introduced by K. Nicholson, where a ring R is called clean if every element is the sum of an idempotent and a unit.

Nicholson proved that every clean ring is an exchange ring, and a ring with central idempotents is clean iff it is an exchange ring. Subsequently, a module M_R is called clean if $End(M_R)$ is a clean ring.

The class of clean rings is quite large and includes, for instance, semiperfect rings, unit-regular rings, strongly-regular rings, and rings of linear transformations of vector spaces.

For the last ten years, the search has been going on to find other interesting classes of clean rings and clean modules. The existence of such classes is closely related to Crawley and Jónsson's open question. Indeed:

It was shown by P. Guil Assensio and A. Srivastava that auto-invariant Modules are clean, and

it was also shown by Camillo, Khurana, Lam, Nicholson and Zhou that every continuous module is clean.

The authors asked: Is a CS module M necessarily clean if it has the finite exchange property?.

While their question still remains open, they provided an affirmative answer when M is quasi-continuous.

Moreover, P. Nielsen showed that a square-free module is clean iff it has the finite exchange property, iff it has the full exchange property.

By modifying and combining the continuity conditions in one single definition and in honor of Y. Utumi, we consider the following new class of modules.

Definition 1 *A right R-module M is called Utumi-module (U-module) if for any two non-zero submodules A and B of M with $A \cong B$ and $A \cap B = 0$, there exist two summands K and L of M such that $A \subseteq^{ess} K$, $B \subseteq^{ess} L$ and $K \oplus L \subseteq^{\oplus} M$. Moreover, a ring R is called right U-ring if the right R-module R_R is a U-module.*

Example 2 *Square-free, quasi-continuous, and automorphism-invariant modules are U-modules.*

Two right R-modules M and N are called orthogonal to each other, if they don't contain non-zero isomorphic submodules.

Theorem 3 *If M is a U-module, then $M = Q \oplus T$ where:*

1. *Q is a quasi-injective module,*
2. *$Q = A \oplus B \oplus D$, where $A \cong B$ and D is isomorphic to a summand of $A \oplus B$,*
3. *T is a square-free module,*
4. *T is Q-injective, and*
5. *Q and T are orthogonal.*

Theorem 4 *A right U-module M is clean if and only if it has the finite exchange property, if and only if it has the full exchange property.*

More on Exchange and Clean Modules

The next definition was introduced in "N. Ding, Y. Ibrahim, M. Yousif, Y. Zhou, $D4$-modules, Journal of Algebra and Its Applications **16**, No. 5 (2017) 1750166 (25 pages)".

Definition 5 *A module M is called dual-square-free (DSF) if M has no proper submodules A and B with $M = A + B$ and $M/A \cong M/B$. The module M is called summand-dual-square-free ($SDSF$) if the submodules A and B are summands of M. A ring R is called right DSF-ring, if it is a DSF-module as a right R-module.*

We should note in passing, being right dual-square-free ring is equivalent to saying that every maximal right ideal of R is two-sided. Rings whose maximal right ideals are two-sided are called quasi-duo, and it is an open question whether every right quasi-duo is left quasi-duo.

Using the work of P. Nielsen in "Square-free modules with the exchange property, *J. Algebra* **323** (2010), 1993-2001", we have the following positive result:

Theorem 6 *Let M be a DSF-module. Then M has the finite exchange property iff M is clean, iff M has the full exchange property.*

Thank You